Conventions and adjunction

DLT-modification.

(X,Δ) not klt.

Proposition

(X,Δ) coefficients of Δ are in $[0,1]$, then $\exists$:

$\pi : Y \to X$ s.t.

1) Y is $\mathbb{Q}$-factorial
2) π only extracts divisors of disc. at most 0.

3) If E (exc. divisors) and Γ the st. transform of Δ.

$$K_Y + \underbrace{E}_{} + \Gamma = \pi^*(K_X + \Delta) + \sum a(F, X, B) F$$

$$a(F, X, B) > 0$$

and $(Y, \Gamma + E)$ is dlt.

4) If (X,Δ) is l.c

Then we can choose π.

$\exists$ divisor support exactly E that is nef over X.

$$\Rightarrow \boxed{K_Y + E + \Gamma = \pi^*(K_X + \Delta).}$$

Proof (of 4)) (X,Δ) is l.c.

We start with a dlt mod.

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta).$$

Now, we can see as $(Y, \Gamma + E)$ is dlt, then

(Y, Γ) is klt, and

Γ is big over X.

By BCHM, we replace (Y, Γ) over X. $(Y, \Gamma + E)$ might not be dlt

l.t. model,

$-E$ is nef over X.

new dlt mod. of $(Y, \Gamma + E)$

$g: W \to Y$ as (Y, Γ) is already klt $\Rightarrow g$ is iso outside of E.

If we look $W \to X$.

$g^*(-E)$ is a nef divisor.

$\square$

DCC sets

Def DCC.

$I \subseteq \mathbb{R}$ has DCC, any decreasing sequence is finite.

Ex. $\boxed{\left\{ 1 - \frac{1}{r} \mid r \in \mathbb{N} \right\}}$

$I \subseteq [0,1]$

Def

$$I_+ := \{0\} \cup \left\{ j \in [0,1] \;\middle|\; j = \sum_{i_p \in I} i_p \right\}$$

Def

$$D(I) = \left\{ a \leq 1 \;\middle|\; a = \frac{m - 1 + f}{m},\ f \in I_+ \right\}$$

Prop. 3.4.1 Let $I \subseteq [0,1]$

1) $D(I)_+ = D(I) \cup \{1\}$

2) $D(D(I)) = D(I) \cup \{1\}$

3) I has DCC iff $D(I)$ has DCC.

4) I has DCC iff $\bar{I}$ has DCC.

Proof

1) $g \in D(I)_+$

$$g = \sum \left(\boxed{\frac{n_i - 1}{n_i}} + \frac{f_i}{n_i} \right) \quad f_i \in I_+$$

at most one $n_i \neq 1$.

$$g = \frac{n-1}{n} + \frac{f}{n} + \sum' f_i$$

$$= \frac{n-1}{n} + \frac{nf + \sum' f_i}{n} \quad \in I_+$$

$\Rightarrow g \in D(I)$

Bounded Pairs

Def. Set $\mathcal{X}$ of varieties

birationally bounded.

$\exists$.

$Z \to T$, finite typ.

$\forall X \in \mathcal{X}$, $\exists t \in T$, s.t.

$\exists f : Z_t \overset{b.r.}{\dashrightarrow} X$.

Def $\mathfrak{D}$ set of log pairs.

log birationally bounded (bounded)

$\exists (Z, B)$ with coeff of B $= 1$.

$Z \to T$. f.t.

$\forall (X, \Delta) \in \mathfrak{D}$, $\exists t \in T$ s.t.

$f : Z_t \overset{b.r.}{\dashrightarrow} X$ (iso. pair).

support of $B_t \supseteq$ st. (Δ) and all exc. div.

Theorem 3.5.2 n, $I \subset [0,1]$

$I \subset \mathbb{Q}$.

I satisfies DCC.

And $\mathfrak{D}_0$ a set of log pairs,

$K_X + \Delta$ is big and coeff $\subseteq I$.

If we have M, k s.t. $\Phi_k(K_X+\Delta)$ is bir. and

$\mathrm{vol}(X, k(K_X+\Delta)) \underline{\leq} M$

Then

$\{\mathrm{Vol}(X, K_X+\Delta)\}$ has DCC.

Reason:

log. bir. bounded.

$\Downarrow$

DCC on the vol.

Def D is potentially birational

X, D big $\mathbb{Q}$-Cartier.

If x and y general pair in X.

we may find $0<\varepsilon<1$

$$0\leq\Delta\sim_{\mathbb{Q}}(1-\varepsilon)D$$

(X,Δ) is not klt at y

(X,Δ) is lc at x an $\{x\}$ is a non-klt centre.

Lemma

(X,D) and D is potentially birational, then

$\phi_{K_X+\lceil D\rceil}$ is birational.

Theorem 3.5.4 (dim $X = n$)

(X, Δ) a klt pair, H ample $\mathbb{Q}$-divisor. If $\exists\ \gamma \geq 1$, and a family $V \to B$ s.t.

- If $x, y \in X$, we can find some $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{Q}} (1-\delta)H$ (some $\delta > 0$)

s.t. $(X, \Delta + \Delta_b)$ is not klt at y and $\exists!$ non-klt place of $(X, \Delta + \Delta_b)$ containing x with center V_b.

$\exists\ D \in W$ normalization of V s.t. Φ_D is bir

$\gamma H|_W - D$ ps. eff.

Then Φ_{mH} is bir

mH is potentially birational

$m = 2p^2\gamma + 1$

$\underset{\dim V.}{p}$

Proof Inductively

$\Delta_K \geq 0$, s.t.

$\Delta_\kappa \sim_{\mathbb{Q}} \lambda H$

$\lambda < 2(p-k)p\gamma + 1$

$(X, \Delta + \Delta_\kappa)$ is lc. at x

not klt at y, and there

is a non-klt center

$Z \subset V_6$ of dim. $\leq k$

containing x.

(induction downwards).

First step.

We $\lambda < 1$, $\lambda := 1 - \delta$:

$(X, \Delta + \underset{\Delta_b}{\underset{\shortparallel}{\Delta_p}})$ satisfies

the condition. by the hypothesis.

Δ_k exists. , we want Δ_{k-1}.

we can assume Z has dimension exactly k.

$Y \subset W$ inverse image of Z.

as ϕ_D is bir..

$$\operatorname{vol}(Y, \gamma H|_Y) \geq \operatorname{vol}(Y, D_Y) \geq 1$$

$$\shortparallel$$

$$\operatorname{vol}(Z, \gamma H|_Z).$$

$$\operatorname{vol}(Z, 2p\gamma H|_V) >$$

$$\boxed{\operatorname{vol}(Z, 2k\gamma H|_V) \geq 2k^k}$$

(By HMX '11)

$\Rightarrow \exists\, \Delta'_k \sim_{\mathbb{Q}} \mu H$

$\mu < 2p\gamma$ s.t. $\exists$ constants

$\underbrace{(X, \Delta + a\Delta_k + b\Delta'_k)}$

is l.c at x, not klt at y

and $\exists$ non-klt center Z'

$x \in Z'$, $\dim(Z') < k$

$a\Delta_k + b\Delta'_k \sim (a\lambda + b\mu)H$

with

$\lambda' = a\lambda + b\mu < 2(p-k)_p\gamma_{+1} + 2p\gamma$

$= 2(p-(k-1))p\gamma_{+1}$

$\underbrace{\Delta_{k-1} := a\Delta_k + b\Delta'_k}_{\Delta_0}$.

$\boxed{\text{Theorem 3.5-5}}$ n,

$\mathcal{B}_0$ set of klt pairs

(X, Δ) klt, $K_X + \Delta$ is amp.

$\Rightarrow$

$\exists\ p, k, l$. $\forall (X, \Delta) \in \mathcal{B}_0$

1) $V \to B$ s.t. if $b \in B$

$\Rightarrow \exists$ $0 \leq \Delta_b \sim_{\mathbb{Q}} (1-\delta) H$ $(\delta > 0)$

s.t. unique non-klt place of $(X, \Delta + \Delta_b)$

with center V_b. ($H := k(K_X + \Delta)$)

There is D on W s.t.

$$W \to V_b$$

ϕ_b is bir. and $lH|_W - D$ is

pseff.

2) $\cdot$ either $p\Delta$ is integral

$\cdot$ $\Delta \subseteq \{1 - \frac{1}{r} \mid r \in \mathbb{N}\}$

Then $\exists$ m integers, t.

$\boxed{\phi_{m\alpha}(K_X+\Delta)}$ is bir. for every $(X,\Delta)\in\mathcal{B}_0$.

Proof) By the previous result, and potentially bir. giving bir. maps.

Maps of form

$$\phi_{K_X + \lceil \underbrace{m}\, H \rceil} \qquad m > m_0$$

is birational.

integer $m_1 > 2m_0$

1)
$$K_X + \lceil \underbrace{(m_1 Kp - 1)}(K_X+\Delta)\rceil$$
$$= K_X + (m_1 K_p - 1)K_X + \lfloor m_1 k_p(\Delta)\rfloor$$
$$= \lfloor m_1 Kp(K_X+\Delta) \rfloor \qquad \square$$

Adjunction:

Lemma 4.1 $(X, \Delta = S + B)$

S has coeff 1.

If S is normal. Then $\exists$

$\Theta = \mathrm{Diff}_S(B)$ on S s.t.:

$(K_X + \Delta)|_S = K_S + \Theta$

1) (X, Δ) is plt $\Rightarrow (S, \Theta)$ is klt

2) (X, Δ) is dlt $\Rightarrow (S, \Theta)$ is dlt.

3) coefficients of Θ belong to $D(\{b_1, \dots, b_r\})$

coeff. of B.

Theorem 4.2 $\dim X = n$.

Let $1 \in I \subseteq [0,1]$.

$X \supseteq V$ subvariety with normalization W.

(X,Δ) and $\Delta' \geq 0$ $\mathbb{R}$-cartier s.t.

1) coeff of $\Delta \in I$

2) (X,Δ) is klt

3) exists unique non-klt place ν for $(X, \Delta+\Delta')$ with centre V.

Then $\exists$ H on W, with coeff

$$\{a \mid 1-a \in LCT_{n-1}(D(I))\} \cup \{1\}$$

s.t. $(K_X+\Delta+\Delta')|_W - K_W + H$ is pseudo-effective.

V covering family of subvar. of X.

$\Psi : U \xrightarrow{\text{res}} W$, Ψ: strict transform H + exceptional divisor.

$$K_U + \Psi \geq (K_X+\Delta)|_U$$

$\boxed{\text{Proof}}$ we have

$(X, \Delta+\Delta')$ is lc, we take dlt mod.

$g: Y \to X$. centre of ν is a divisor S on Y.

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ W \arrow[r] & X \end{tikzcd}$$

$$K_X + \underline{S} + \underline{\Gamma} = g^*(K_X+\Delta) + \underline{\underline{E}}$$

strict. transform of Δ and other exc.

as (X, Δ) is klt

$\Rightarrow$ $E \geq 0$.

$$K_X + S + \Gamma + \underbrace{\Gamma'}_{\text{strict transform of } \Delta'} = g^*(K_X + \Delta + \Delta')$$

$(Y, S+\Gamma+\Gamma')$ is dlt by def. and $\Rightarrow (Y, S+\Gamma)$ is also dlt.

By adj.:

$$(K_Y+S+\Gamma)|_S = K_S + \Phi$$

$$(K_Y+S+\Gamma+\Gamma')_S = K_S + \Phi'$$

Γ, Γ' will also have coeff in I

Φ, Φ' have coeff. in $D(I)$.

B a prime divisor on W,

$$\mu = LCT_B(S, \Phi, f^*B)$$

$$\sup(t, (S, \Phi + tf^*B)$$

$$\lambda = LCT_B(S, \Phi', f^*B)$$

Θ s.t.

$\underbrace{\mathrm{mult}_B(\Theta) = 1 - \mu}$

Θ_b s.t.

$= \underbrace{\mathrm{mult}_B(\Theta') = 1 - \lambda}.$

Θ have coeff. as we want.

as $\Gamma' \geq 0$, $\Phi \leq \Phi'$

$\lambda \geq \mu$

$\Theta \leq \Theta_b$.

Θ_b is the divisor in Kawamata subadjunction:

$(K_X + \Delta + \Delta')|_W - (K_W + \Theta)$

still is ps.eff.

here we we have the coefficients.

$\square$